

ON SCHRÖDINGER'S EQUATION, 3-DIMENSIONAL BESSEL BRIDGES, AND PASSAGE TIME PROBLEMS

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ABSTRACT. In this work we relate the density of the first-passage time of a Wiener process to a moving boundary with the three dimensional Bessel bridge process and a solution of the heat equation with a moving boundary. We provide bounds.

1. THE PROBLEM

Problem 1.1. *The main motivation of this work is to find the density of the first time T , that a one-dimensional, standard Brownian motion B , reaches the moving boundary f :*

$$(1) \quad T := \inf \{t \geq 0 | B_t = f(t)\}$$

where $f(t) := a + \int_0^t f'(u)du$, $f''(t) > 0$, and $\int_0^t (f'(u))^2 du < \infty$ for all $t > 0$.

Alternatively, assuming the previous assumptions:

Proposition 1.2. *Given that*

$$(2) \quad h(s, a) := \frac{|a|}{\sqrt{2\pi s^3}} \exp \left\{ -\frac{a^2}{2s} \right\} \quad s \geq 0, \quad a \in \mathbb{R}$$

is the density of the first time that B reaches the fixed level a , the process \tilde{X} is a 3-dimensional Bessel bridge, which has the following dynamics:

$$d\tilde{X}_t = dW_t + \left(\frac{1}{\tilde{X}_t} - \frac{\tilde{X}_t}{s-t} \right) dt, \quad \tilde{X}_0 = a, \quad t \in [0, s).$$

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(Where W is a Wiener process.) Then, the distribution of T equals

$$(3) \quad \begin{aligned} \mathbb{P}(T < t) &= \int_0^t \tilde{\mathbb{E}} \left[\exp \left\{ - \int_0^s f''(u) \tilde{X}_u du \right\} \right] \\ &\quad \times \exp \left\{ - \frac{1}{2} \int_0^s (f'(u))^2 du - f'(0)a \right\} h(s, a) ds \end{aligned}$$

Proof. From Girsanov's theorem, the fact that the boundary f is twice continuously differentiable, the following Radon-Nikodym derivative:

$$\begin{aligned} \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}(t) &:= \exp \left\{ - \int_0^t f'(s) d\tilde{B}_s - \frac{1}{2} \int_0^t (f'(s))^2 ds \right\} \\ &:= \exp \left\{ -f'(t)\tilde{B}_t + \int_0^t f''(s)\tilde{B}_s ds - \frac{1}{2} \int_0^t (f'(s))^2 ds \right\}, \end{aligned}$$

is indeed a *martingale* [Novikov's condition] and induces the following relationship:

$$\begin{aligned} \mathbb{P}(T \leq t) &:= \tilde{\mathbb{E}} \left[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}(t) \mathbb{I}_{(T \leq t)} \right] \\ &= \tilde{\mathbb{E}} \left[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}(T) \mathbb{I}_{(T \leq t)} \right] \\ &= \int_0^t \tilde{\mathbb{E}} \left[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}(T) \middle| T = s \right] h(s, a) ds, \end{aligned}$$

where the second equality follows from the optional sampling theorem, and the third from conditioning with respect to T under \tilde{P} , and h is defined in (2).

In Revuz and Yor (1999) (Chapter 11), it is shown that calculating:

$$(4) \quad \tilde{\mathbb{E}} \left[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}(T) \middle| T = s \right]$$

is equivalent to finding the expected value of a functional of a 3-dimensional Bessel bridge \tilde{X} , which at time 0 starts at a and at time

s equals 0. Indeed,

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} (T) \middle| T = s \right] \\
&= \tilde{\mathbb{E}} \left[\exp \left\{ -f'(T)a + \int_0^T f''(u) \tilde{B}_u du \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \int_0^T (f'(u))^2 du \right\} \middle| T = s \right] \\
&= \tilde{\mathbb{E}} \left[\exp \left\{ -f'(s)a + \int_0^s f''(u)(a - \tilde{X}_u) du - \frac{1}{2} \int_0^s (f'(u))^2 du \right\} \right] \\
&= \tilde{\mathbb{E}} \left[\exp \left\{ -f'(s)a + (f'(s) - f'(0))a - \int_0^s f''(u) \tilde{X}_u du \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \int_0^s (f'(u))^2 du \right\} \right] \\
(5) \quad &= \tilde{\mathbb{E}} \left[\exp \left\{ - \int_0^s f''(u) \tilde{X}_u du \right\} \right] \\
&\quad \times \exp \left\{ - \frac{1}{2} \int_0^s (f'(u))^2 du - f'(0)a \right\}
\end{aligned}$$

thus the process $a - \tilde{X}$ equals 0 at $t = 0$, and at s reaches level a for the first time, as required [see Figures 1–3]. \square

From equation (5) it is clear that our next goal is to compute the following expected value:

$$(6) \quad \tilde{\mathbb{E}} \left[\exp \left\{ - \int_0^s f''(u) \tilde{X}_u du \right\} \right].$$

Theorem 1.3. *Suppose that $v(t, a) : [0, s] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous, is of class $\mathbb{C}^{1,2}$ and satisfies the Cauchy problem*

$$\begin{aligned}
(7) \quad -\frac{\partial v}{\partial t} + f''(t)av &= \frac{1}{2} \frac{\partial^2 v}{\partial a^2} + \left(\frac{1}{a} - \frac{a}{s-t} \right) \frac{\partial v}{\partial a} & [0, s] \times \mathbb{R}^+, \\
v(s, a) &= 1, & a \in \mathbb{R}^+,
\end{aligned}$$

as well as

$$0 \leq v(t, a) \leq 1 \quad \forall \, t, a \in \mathbb{R}^+$$

Then $v(t, a)$ admits the stochastic representation

$$v(t, a) = \mathbb{E}^{t,a} \left[\exp \left\{ - \int_t^s f''(u) \tilde{X}_u du \right\} \right]$$

Proof. We proceed as in the proof of Theorem 5.7.6, pp. 366–367, Karatzas & Shreve (1991). Applying Ito's rule to the process

$$v(y, X_y) \exp \left\{ - \int_y^t f''(u) \tilde{X}_u du \right\},$$

$y \in [t, s]$, and obtain, with $\tau_n := \inf\{t \leq y \leq s \mid \tilde{X}_y \geq n\}$,

$$\begin{aligned} v(t, a) &= \tilde{\mathbb{E}}^{t,a} \left[\exp \left\{ - \int_t^s f''(u) \tilde{X}_u du \right\} \mathbb{I}_{(\tau_n > s)} \right] \\ &\quad + \tilde{\mathbb{E}}^{t,a} \left[v(\tau_n, X_{\tau_n}) \exp \left\{ - \int_t^{\tau_n} f''(u) \tilde{X}_u du \right\} \mathbb{I}_{(\tau_n \leq s)} \right] \end{aligned}$$

The second term converges to zero as $n \rightarrow \infty$, since

$$\begin{aligned} &\tilde{\mathbb{E}}^{t,a} \left[v(\tau_n, X_{\tau_n}) \exp \left\{ - \int_t^{\tau_n} f''(u) \tilde{X}_u du \right\} \mathbb{I}_{(\tau_n \leq s)} \right] \\ &\leq \tilde{\mathbb{E}}^{t,x} \left[v(\tau_n, \tilde{X}_{\tau_n}) \mathbb{I}_{(\tau_n \leq s)} \right] \\ &\leq \tilde{\mathbb{P}}^{t,x}(\tau_n \leq s) \\ &= \tilde{\mathbb{P}}^{t,x} \left(\max_{t \leq \theta \leq s} \tilde{X}_\theta \geq n \right) \\ &\leq \frac{\tilde{\mathbb{E}}^{t,x} \left[\max_{t \leq \theta \leq s} \tilde{X}_\theta^{2m} \right]}{n^{2m}} \end{aligned}$$

[see Pitman and Yor (1998) for the moments of the running maximum of \tilde{X}]. Finally, the first term converges to

$$\tilde{\mathbb{E}}^{t,a} \left[\exp \left\{ - \int_t^s f''(u) \tilde{X}_u du \right\} \right]$$

either by the dominated or by the monotone convergence theorem. \square

2. SOLUTIONS OF EQUATION (7)

Proposition 2.1. *Solutions to (7) are of the following form*

$$v(t, a) = \frac{w(t, a)}{h(s - t, a)}$$

where

$$(8) \quad -w_t(t, a) + f''(t)aw(t, a) = \frac{1}{2}w_{aa}(t, a).$$

and $h(s, a)$ as in (2).

Proof. Setting

$$u(t, a) = 1/h(s - t, a)$$

and $v(t, a) = u(t, a)w(t, a)$, we have that

$$\begin{aligned} u_t(t, a) &= \left[\frac{a^2}{2(s - t)^2} - \frac{3}{2(s - t)} \right] u(t, a) \\ u_a(t, a) &= - \left[\frac{1}{a} - \frac{a}{s - t} \right] u(t, a) \\ u_{aa}(t, a) &= \left[\frac{2}{a^2} + \frac{a^2}{(s - t)^2} - \frac{1}{s - t} \right] u(t, a) \end{aligned}$$

and

$$\begin{aligned} v_t &= u_t w + u w_t & v_x &= u_x w + u w_x \\ v_{xx} &= u_{xx} w + 2u_x w_x + u w_{xx}. \end{aligned}$$

Hence from (7) and (8) it follows that

$$\left[-u_t - \frac{1}{2}u_{aa} - \left(\frac{1}{a} - \frac{a}{s - t} \right) u_a \right] w = \left[u_a + \left(\frac{1}{a} - \frac{a}{s - t} \right) u \right] w_a,$$

as claimed. \square

Theorem 2.2. *Solutions to (8) are given by*

$$\begin{aligned} (9) \quad w(t, a) &= e^{\frac{1}{2} \int_t^s (f'(u))^2 du + a f'(t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi(y) e^{-\frac{1}{2} y^2 (s - t) + i y (a + \int_t^s f'(u) du)} dy \\ &= e^{\frac{1}{2} \int_t^s (f'(u))^2 du + a f'(t)} \omega(s - t, a + \int_t^s f'(u) du) \end{aligned}$$

where ω is a solution to the heat equation.

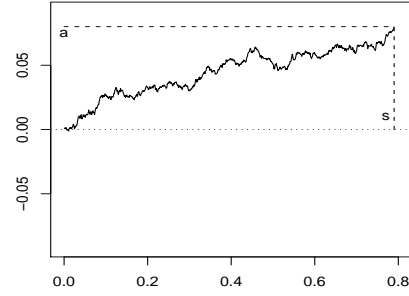


FIGURE 1. (a) Brownian motion conditioned to hit level a for the first time at time s .

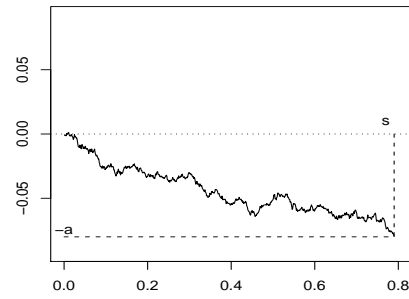


FIGURE 2. (b) Reflection of the conditioned process.

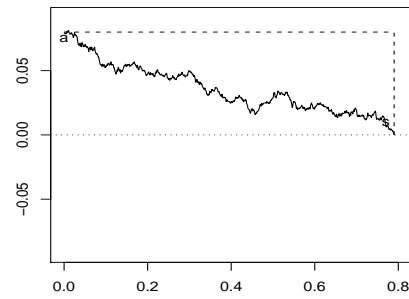


FIGURE 3. (c) Three-dimesional Bessel bridge \tilde{X}

Proof. Let

$$(10) \quad \hat{w}(t, \lambda) := \int_{-\infty}^{\infty} e^{-i\lambda a} w(t, a) da.$$

Applying the Fourier transform to (8) we have

$$-\hat{w}_t(t, \lambda) + i f''(t) \hat{w}_\lambda(t, \lambda) + \frac{1}{2} \lambda^2 \hat{w}(t, \lambda) = 0 \quad i := \sqrt{-1}.$$

Next, set $y = \lambda + i f'(t)$ and $\hat{w}(t, \lambda) = \tilde{w}(t, y)$, that is:

$$\hat{w}_t = \tilde{w}_t + i f''(t) \tilde{w}_y, \quad \hat{w}_\lambda = \tilde{w}_y$$

which after substitution in (10) leads to

$$-\tilde{w}_t(t, y) + \frac{1}{2} (y - i f'(t))^2 \tilde{w}(t, y) = 0.$$

Consequently

$$\begin{aligned} \tilde{w}(t, y) &= \Pi(y) \exp \left\{ -\frac{1}{2} \int_t^s (y - i f'(u))^2 du \right\} \\ &= \Pi(y) \exp \left\{ -\frac{1}{2} y^2 (s - t) + i y \int_t^s f'(u) du + \frac{1}{2} \int_t^s (f'(u))^2 du \right\} \end{aligned}$$

which alternatively implies that

$$\begin{aligned} w(t, a) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi(y) e^{-\frac{1}{2} y^2 (s-t) + i y \int_t^s f'(u) du + \frac{1}{2} \int_t^s (f'(u))^2 du} e^{i \lambda a} d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi(y) e^{-\frac{1}{2} y^2 (s-t) + i y \int_t^s f'(u) du + \frac{1}{2} \int_t^s (f'(u))^2 du} e^{i y a + a f'(t)} dy \\ &= e^{\frac{1}{2} \int_t^s (f'(u))^2 du + a f'(t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi(y) e^{-\frac{1}{2} y^2 (s-t) + i y (a + \int_t^s f'(u) du)} dy \end{aligned}$$

as claimed. \square

Theorem 2.3. *The density ω of the first passage time T defined in (1), is bounded by*

$$\begin{aligned} h(s, a) e^{-a f'(0) - \frac{1}{2} \int_0^s (f'(u))^2 du - \int_0^s f''(u) \mathbb{E}^a(\tilde{X}_u) du} \\ \leq \omega \left(s, a + \int_0^s f'(u) du \right) \\ \leq h(s, a) e^{-a f'(0) - \frac{1}{2} \int_0^s (f'(u))^2 du} \end{aligned}$$

where ω is a solution of the forward heat equation and h is as in (2).

Proof. It follows from Jensen's inequality,

$$\exp \left\{ - \int_0^s f''(u) \tilde{\mathbb{E}}^{t,a}(\tilde{X}_u) du \right\} \leq \tilde{\mathbb{E}}^{t,a} \left[\exp \left\{ - \int_0^s f''(u) \tilde{X}_u du \right\} \right] \leq 1$$

and equations (8) and (9). \square

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